Taylor ~~Swift~~ Series (Maclaurin Series)

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Outline

What is it?

- Approximation of a function with an infinite series
- Approximates near $x = 0$

Why?

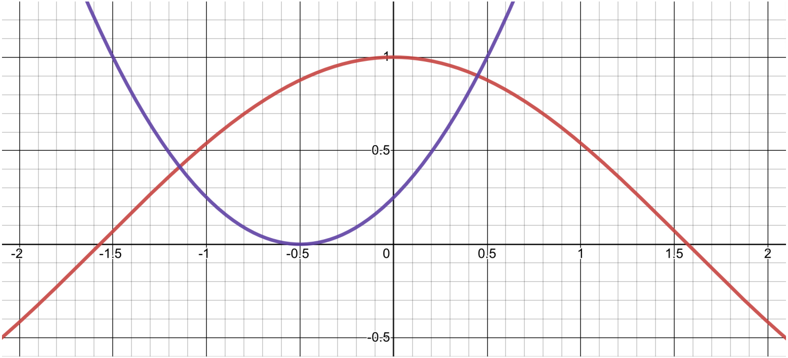
- To compute $\sin x$, $\cos x$, and e^x *fast*
- Calculators (your TI) use this technique
- To simplify equations/functions
- In simple pendulum, we *approximated* $\sin x$ with x

Derivation

- Calculators can multiply, add, subtract, divide, and take powers of whole numbers *quickly*
- Using *polynomials* will be efficient
- Since polynomials are just multiplications, additions, and exponentiations of numbers

Derivation

Figure: The Function $\cos x$



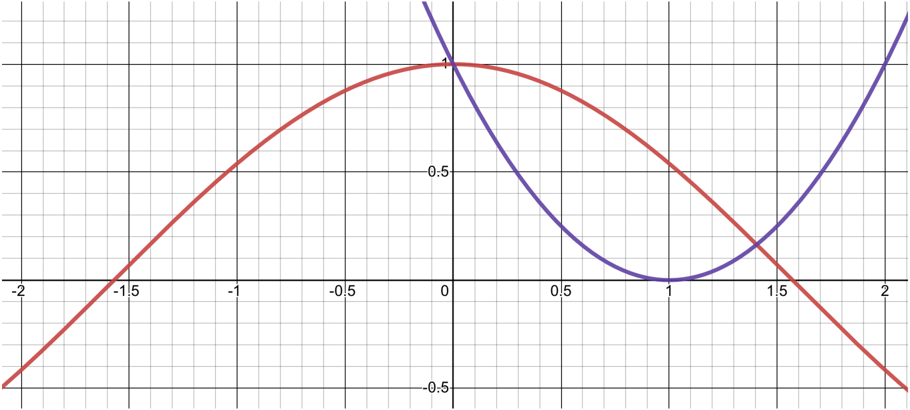
- Approximate to two degrees
- Find real numbers for c_0, c_1, and c_2 that approximate $\cos x$ the *best*

$$\cos x \approx c_0 + c_1 x + c_2 x^2$$

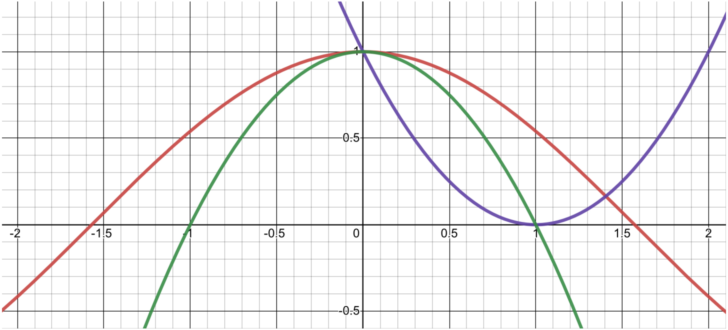
Derivation

- Approximation *near* $x = 0$

$$\cos 0 = c_0 + c_1 \cdot 0 + c_2 \cdot 0^2$$
$$c_0 = 1$$



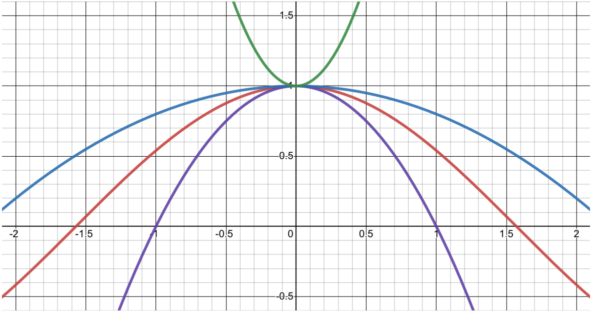
Derivation



- The green function is better, but why?
- The rate of change is the same as $\cos x$ at $x = 0$
- Approximation must have the same rate of change at $x = 0$
- $\cos'(x) = -\sin x$, and $(c_0 + c_1 x + c_2 x^2)' = c_1 + 2c_2 x$

$$-\sin 0 = 0 = c_1 + 2c_2 \cdot 0$$
$$c_1 = 0$$

Derivation



- $\cos x$ *curves* downwards at $x = 0$
- So, the second derivative is negative
- So, the rate of change is decreasing
- Same second derivative will ensure that they curve at the same rate

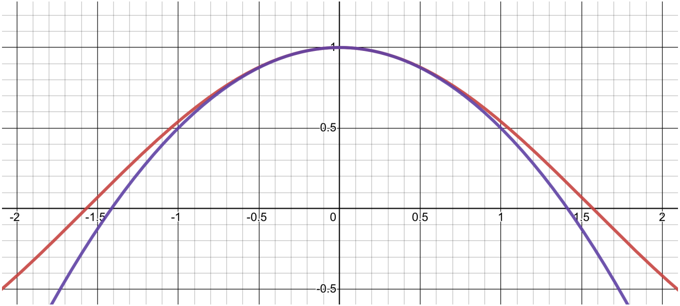
$$\cos''(x) = -\cos x$$

$$\left(c_0 + c_1 x + c_2 x^2\right)'' = 2c_2$$

Derivation

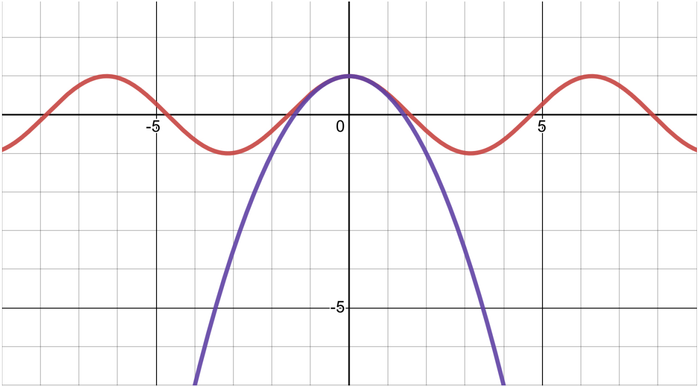
- $\cos''(x) = -\cos x$, and $\left(c_0 + c_1 x + c_2 x^2\right)'' = 2c_2$

$$-\cos 0 = 2c_2$$
$$-1 = 2c_2$$
$$c_2 = -\frac{1}{2}$$
$$\cos x \approx 1 - \frac{1}{2}x^2$$



Derivation

- Okay, but how *good* is the approximation?
- For $x = 0.1$, $\cos x = 0.99500417$, and the approximation, $1 - \frac{1}{2}x^2 = 0.995$
- For $x = 0.25$, $\cos x = 0.9689124$, and the approximation, $1 - \frac{1}{2}x^2 = 0.96875$



The More the Merrier

- But why stop at x^2? Why not go further?
- More terms will give more *control* over the approximation
- Add another term c_3x^3 to the approximation

$$\cos x \approx 1 - \frac{1}{2}x^2 + c_3x^3$$

- Taking the third derivative of a polynomial, all the terms that have a power less than 3 will vanish
- And, $\cos'''(x) = \sin x$
- Taking the derivative,

$$\cos'''(x) = \sin x = \left(-x + 3c_3x^2\right)'' = \left(-1 + 2 \cdot 3c_3x\right)' = 1 \cdot 2 \cdot 3 \cdot c_3$$

$$\sin 0 = 1 \cdot 2 \cdot 3 \cdot c_3$$

$$c_3 = 0$$

The More the Merrier

$$\cos x \approx 1 - \frac{1}{2}x^2$$

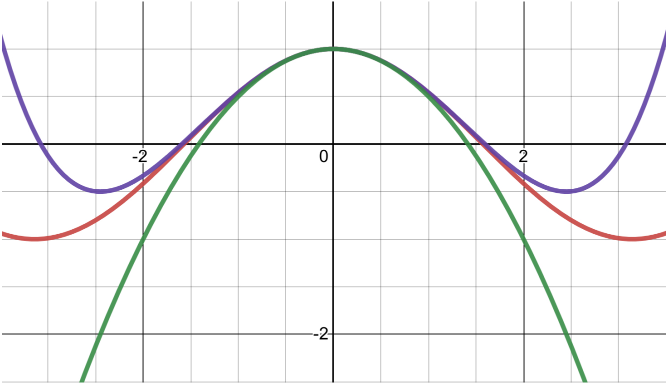
- This approximation is the best for all cubic polynomials, as well as all the quadratic polynomials
- But, we can do better if we extend to another term

$$\cos x \approx 1 - \frac{1}{2}x^2 + c_4 x^4$$

$$\begin{aligned} \cos^{(4)}(x) &= \cos x \\ \left(1 - \frac{1}{2}x^2 + c_4 x^4\right)^{(4)} &= 1 \cdot 2 \cdot 3 \cdot 4 \cdot c_4 \\ \cos 0 &= 1 \cdot 2 \cdot 3 \cdot 4 \cdot c_4 \\ c_4 &= \frac{1}{24} \end{aligned}$$

The More the Merrier

$$\cos x \approx 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4$$



- This is a really good approximation of $\cos x$
- For most physics problems, this would be fine
- But, we are dealing with maths

Notice a Few Things

- Firstly, factorials come up quite naturally from taking n successive derivatives of $c_n x^n$

$$\frac{\mathrm{d}\left(c_n x^n\right)}{\mathrm{d}x} = n \cdot c_n \cdot x^{n-1}$$
$$\frac{\mathrm{d}^2\left(c_n x^n\right)}{\mathrm{d}x^2} = n \cdot (n-1) \cdot c_n \cdot x^{n-2}$$
$$\frac{\mathrm{d}^3\left(c_n x^n\right)}{\mathrm{d}x^3} = n \cdot (n-1) \cdot (n-2) \cdot c_n \cdot x^{n-3}$$
$$\vdots$$
$$\frac{\mathrm{d}^n\left(c_n x^n\right)}{\mathrm{d}x^n} = n! \cdot c_n$$

- So, we have to divide by the appropriate factorial to cancel out this effect

$$c_n = \frac{\text{desired derivative value}}{n!}$$

Notice a Few Things

- Secondly, adding new terms does *not* mess up older terms
- Other higher-order terms that have x will not affect the lower order terms

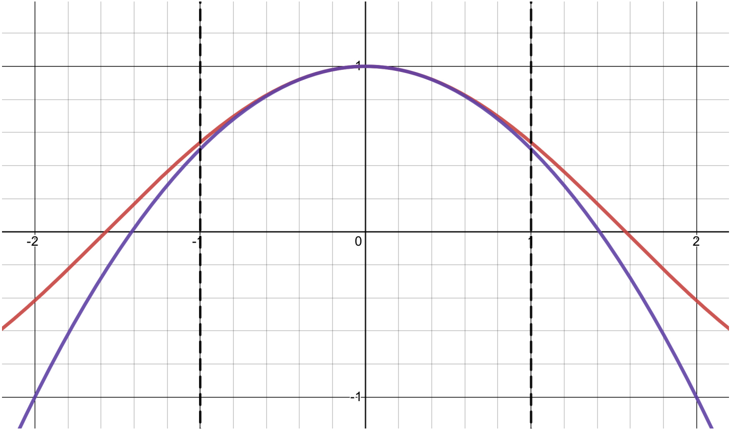
$$P(x) = 1 - \frac{1}{2}x^2 + c_4 x^4$$
$$P''(0) = 2\left(-\frac{1}{2}\right) + 3 \cdot 4(0)^2$$

- Each derivative of a polynomial at $x = 0$ is controlled by one and only one of the coefficients

$$P(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4$$

Notice a Few Things

- Derivative information at $x = 0 \longrightarrow$ output information near $x = 0$



Notice a Few Things

$$\begin{aligned}\cos 0 &= 1\\ \cos' 0 &= 0\\ \cos'' 0 &= -1\\ \cos''' 0 &= 0\\ \cos^{(4)} 0 &= 1\\ &\vdots\end{aligned}$$

$$P(x) = 1 + 0\frac{x^1}{1!} + -1\frac{x^2}{2!} + 0\frac{x^3}{3!} + 1\frac{x^4}{4!} + \cdots$$

- Those factorials are there to cancel out the cascading effect of derivatives

Maclaurin Series

- The same approach can be used for *any* function
- We can approximate $f(x)$ near $x = 0$ with any degree of accuracy we want

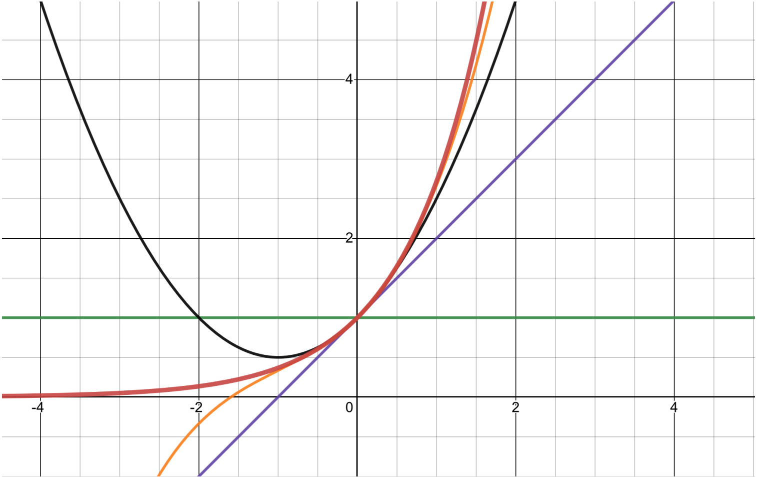
$$P(x) = f(0) + f'(0)\frac{x}{1} + f''(0)\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)x^n}{n!}$$

- This summation at infinity is the *Maclaurin series* of $f(x)$
- Let us approximate the function e^x (which is in the *Formula Booklet*)

Maclaurin Series

- Any derivative of e^x is e^x, so $e^0 = 1$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$



Euler's Formula

- We can, in fact, use this to prove

$$\cos\theta + i\sin\theta = \mathrm{e}^{i\theta}$$

$$\begin{aligned}
\cos\theta &= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots \\
\sin\theta &= \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots \\
e^{i\theta} &= 1 + (i\theta) + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \frac{(i\theta)^6}{6!} + \frac{(i\theta)^7}{7!} + \cdots \\
&= 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} - \frac{\theta^6}{6!} - \frac{i\theta^7}{7!} + \cdots \\
&= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots\right) \\
&= \cos\theta + i\sin\theta \\
&= \operatorname{cis}\theta
\end{aligned}$$

Example 1

- Find the Maclaurin series of the function $f(x) = \mathrm{e}^x \sin x$ up to the term x^3
- Two methods: multiply series expansions of e^x and $\sin x$, or rigour

$$\begin{aligned} f(x) &= \mathrm{e}^x \sin x \\ f'(x) &= \mathrm{e}^x \sin x + \mathrm{e}^x \cos x \\ f''(x) &= \mathrm{e}^x \cos x - \mathrm{e}^x \sin x + \mathrm{e}^x \sin x + \mathrm{e}^x \cos x = 2e^x \cos x \\ f'''(x) &= 2\mathrm{e}^x \cos x - 2\mathrm{e}^x \sin x = 2\mathrm{e}^x(\cos x - \sin x) \end{aligned}$$

- $f(0) = 0,\ f'(0) = 1,\ f''(0) = 2,\ f'''(0) = 2$

$$\begin{aligned} f(x) &= 0 + 1x + \frac{2x^2}{2!} + \frac{2x^3}{3!} \\ &= x + x^2 + \frac{x^3}{3} \end{aligned}$$

Example 2

- Find the Maclaurin series of the function $f(x) = (1+x)^p$ for $p \in \mathbb{R}$

$$
\begin{aligned}
f(x) &= (1+x)^p; f(0) = 1 \\
f'(x) &= p(1+x)^{p-1}; f'(0) = p \\
f''(x) &= p(p-1)(1+x)^{p-2}; f''(0) = p(p-1) \\
&\vdots \\
f^{(n)}(x) &= p(p-1)(p-2)\cdots(p-n+1)(1+x)^{p-n}; \\
f^{(n)}(0) &= p(p-1)(p-2)\cdots(p-n+1)
\end{aligned}
$$

$$
\begin{aligned}
P(x) &= 1 + px + \frac{p(p-1)x^2}{2!} + \frac{p(p-1)(p-2)x^3}{3!} + \cdots \\
&= \sum_{n=0}^{\infty} \frac{p(p-1)(p-2)\cdots(p-n+1)x^n}{n!} \\
&= \sum_{n=0}^{\infty} \binom{p}{n} x^n
\end{aligned}
$$

Connection to Taylor Series

- Maclaurin series approximates a function near $x = 0$
- Can be approximated near any point $x = a$ using Taylor series

$$P(x) = f(a) + f'(a)(x-a) + f''(a)\frac{(x-a)^2}{2!} + f'''(a)\frac{(x-a)^3}{3!} + \cdots$$

Figure: The Function $\ln x$ and Approximation